PERGAMON

International Journal of Solids and Structures 39 (2002) 1199–1211

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Elastic study on singularities interacting with interfaces using alternating technique Part II. Isotropic trimaterial

S.T. Choi, Y.Y. Earmme *

Department of Mechanical Engineering, ME3013, Korea Advanced Institute of Science and Technology, Science Town, Daejon 305-701, South Korea

Received 19 March 2001

Abstract

Singularity problems in an isotropic trimaterial are analyzed by the same procedure as in an anisotropic trimaterial of Part I [Int. J. Solids Struct. 39, 943–957]. 'Trimaterial' denotes an infinite body composed of three dissimilar materials bonded along two parallel interfaces. Linear elastic isotropic materials under plane deformations are assumed, in which the plane of deformation is perpendicular to the two parallel interface planes, and thus Muskhelishvili's complex potentials are used. The method of analytic continuation is alternatively applied across the two parallel interfaces in order to derive the trimaterial solution in a series form from the corresponding homogeneous solution. A variety of problems, e.g. a bimaterial (including a half-plane problem), a finite thin film on semi-infinite substrate, and a finite strip of thin film, etc, can be analyzed as special cases of the present study. A film/substrate structure with a dislocation is exemplified to verify the usefulness of the solutions obtained.

1. Introduction

Advances in thin film and layered structure technology have served as a driving force for the evolution of electronic and opto-electronic devices. To make electronic systems reliable, the analysis of defects, which are inevitable and affect the performance of the systems, have attracted much attention of many researchers (Freund, 2000). For example, elastic study on misfit dislocations in strained epitaxial films offers some useful results such as a critical thickness of epitaxial films (Freund, 1993). The solution of dislocations in thin film and layered structures are also used to simulate cracks in the structures by continuous distributions of dislocations (Suo and Hutchinson, 1989a,b; Fleck et al., 1991; Erdogan and Wu, 1993). From the mechanical point of view, these dislocations are treated as singularities and the analysis of the elastic field near the singularities plays an important role in understanding the behavior of the structures.

* Corresponding author. Tel.: +82-42-869-3013; fax: +82-42-869-3210.
E-mail address: yyearmme@kaist.ac.kr (Y.Y. Earmme).

PII: S0020-7683(01)00231-1

By using Muskhelishvili's complex potentials (Muskhelishvili, 1953) and the method of analytic continuation, Suo (1989) expressed the solution for a singularity in an isotropic bimaterial in terms of that for the same singularity in a homogeneous isotropic medium. The elastic fields due to dislocations in an isotropic strip (i.e., homogeneous material), film/substrate structure (i.e., bimaterial), or layered material (i.e., trimaterial), are studied by using complex potentials in conjunction with the Fourier integral transform by many researchers (Lee and Dundurs, 1973; Suo and Hutchinson, 1989a,b; Fleck et al., 1991; Erdogan and Wu, 1993; Zhang, 1995). But their method requires the inverse Fourier transform procedure, which is somewhat cumbersome. It will be shown later that the above works can be dealt with as special cases of the present study, which makes the present method versatile.

In this study, we employ the same procedure as in Part I (Choi and Earmme, in press). In other words, the method of analytic continuation (Suo, 1989) is alternatively applied to the two parallel interfaces to solve singularity problems in an isotropic trimaterial, in which the homogeneous solution for the same singularities is used as a base. Chao and Kao (1997) analyzed an isotropic trimaterial under an anti-plane concentrated force through iterations of Möbius transformation. Their method is similar to the alternating technique and their solution coincides with the result of the present study as shown in Appendix A. In Section 2, we briefly study the theory of isotropic elasticity, and then, Sections 3–5 are devoted to a singularity in a homogeneous medium, a bimaterial, and a trimaterial, respectively. The convergence of the trimaterial solution, the energetic forces on a dislocation, and an example are dealt with in Section 6. Section 7 concludes this article.

2. Isotropic elasticity

The components of the stresses and displacements for an isotropic body under plane deformation are expressed in terms of two complex potentials $\Phi(z)$ and $\Omega(z)$ as follows (Muskhelishvili, 1953):

$$\sigma_{11} + \sigma_{22} = 2\left[\Phi(z) + \overline{\Phi(z)}\right], \tag{1}$$

$$\sigma_{22} + \mathrm{i}\sigma_{12} = \overline{\Phi(z)} + \Omega(z) + (\bar{z} - z)\Phi'(z), \tag{2}$$

$$-2\mathrm{i}G\frac{\partial}{\partial x_1}(u_2 + \mathrm{i}u_1) = \kappa\overline{\Phi(z)} - \Omega(z) - (\bar{z} - z)\Phi'(z), \tag{3}$$

where $\kappa = 3 - 4\nu$ for plane strain and $(3 - \nu)/(1 + \nu)$ for plane stress, ν and G are Poisson's ratio and shear modulus, respectively. Here the overbar $\overline{(\;)}$ represents the complex conjugate and the prime $(\;)'$ the derivative with respect to $z = x_1 + \mathrm{i}x_2$. Since the anti-plane problem can be separated from the in-plane problem, the case of the anti-plane problem is dealt with in Appendix A in an analogous way to the procedure described in this study for in-plane problem.

When an isotropic elastic bimaterial is in a state of plane deformation and loaded by prescribed surface tractions, and there are no net forces on the internal boundaries, the stresses depend on only two non-dimensional Dundurs parameters (Dundurs, 1969),

$$\alpha_{\mathrm{ab}} = \frac{G_{\mathrm{a}}(\kappa_{\mathrm{b}} + 1) - G_{\mathrm{b}}(\kappa_{\mathrm{a}} + 1)}{G_{\mathrm{a}}(\kappa_{\mathrm{b}} + 1) + G_{\mathrm{b}}(\kappa_{\mathrm{a}} + 1)}, \qquad \beta_{\mathrm{ab}} = \frac{G_{\mathrm{a}}(\kappa_{\mathrm{b}} - 1) - G_{\mathrm{b}}(\kappa_{\mathrm{a}} - 1)}{G_{\mathrm{a}}(\kappa_{\mathrm{b}} + 1) + G_{\mathrm{b}}(\kappa_{\mathrm{a}} + 1)}, \tag{4}$$

where a and b refer to the two materials composing the bimaterial. A parallelogram enclosed by $\alpha = \pm 1$ and $\alpha - 4\beta = \pm 1$ in the (α, β) plane is admissible for the physical values of α and β as shown in Fig. 1, in which

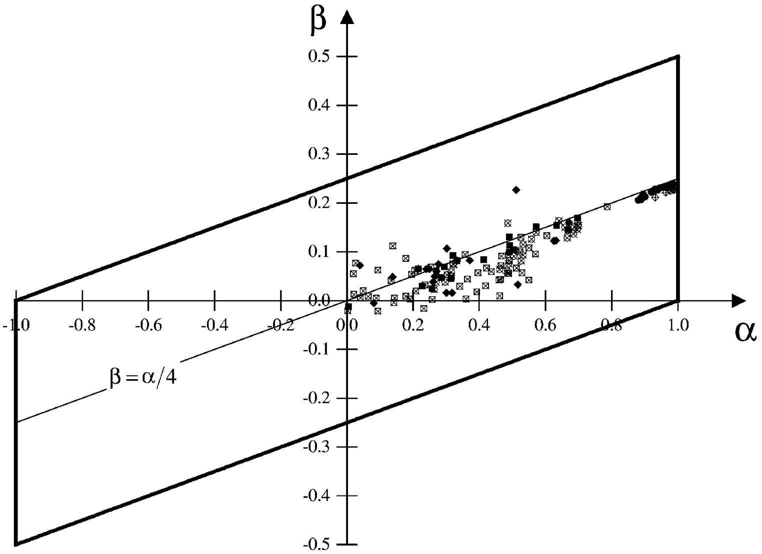

Fig. 1. Dundurs parameters α and β for typical material combinations (Suga et al. 1988).

the values of α and β for various material combinations are also plotted (Suga et al., 1988). The two parameters stand for the mismatch of elastic constants between a and b materials. Noting that the choice of the two parameters to be formed from the elastic constants is not unique, another pairs are defined as

$$\Lambda_{ab} = \frac{\alpha_{ab} + \beta_{ab}}{1 - \beta_{ab}}, \qquad \Pi_{ab} = \frac{\alpha_{ab} - \beta_{ab}}{1 + \beta_{ab}}, \tag{5}$$

which are more convenient for our purpose. The parallelogram in Fig. 1 is transformed into a region enclosed by $\Pi = 1/\Lambda$, $\Pi = (\Lambda - 1)/(\Lambda + 3)$, and $\Pi = (2\Lambda + 1)/(\Lambda + 2)$ in (Λ, Π) plane as shown in Fig. 2. Some discussions related to Λ and Π will be presented in connection with the convergence of series solutions in Section 6.1.

3. A singularity in a homogeneous medium

When a singularity is in an infinite homogeneous isotropic material, the solutions $\Phi_0(z)$ and $\Omega_0(z)$ are as follows (Muskhelishvili, 1953; Suo, 1989):

$$\Phi_0(z) = -\frac{Q}{z - s}, \qquad \Omega_0(z) = -\frac{Q(\bar{s} - s)}{(z - s)^2} + \frac{\overline{Q}\hat{\kappa}}{z - s}, \tag{6}$$

where $s = x_1^0 + \mathrm{i}x_2^0$ is the position of the singularity, and Q and $\hat{\kappa}$ are defined as

(i) *for a point force* $P(= P_x + \mathrm{i}P_y)$

$$Q = \frac{P_x + \mathrm{i}P_y}{2\pi(\kappa + 1)}, \quad \hat{\kappa} = \kappa, \tag{7}$$

(ii) *for an edge dislocation* $b(= b_x + \mathrm{i}b_y)$

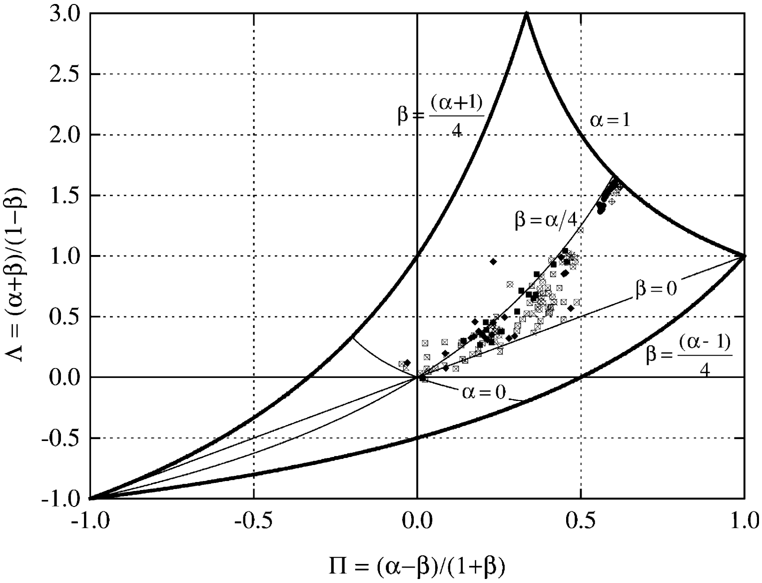

Fig. 2. Bimaterial constants Π and Λ for typical material combinations.

$$Q = \frac{G\mathrm{i}(b_x + \mathrm{i}b_y)}{\pi(\kappa + 1)}, \quad \hat{\kappa} = -1. \tag{8}$$

These fields will be used for the corresponding problems of the same singularity in a bimaterial and a trimaterial in the following sections.

4. A singularity in a bimaterial and the method of analytic continuation

The solution of a singularity in a bimaterial bonded along x_1-axis as shown in Fig. 3 is constructed by the method of analytic continuation in terms of the homogeneous solution $\Phi_0(z)$ and $\Omega_0(z)$ (Suo, 1989). First, a

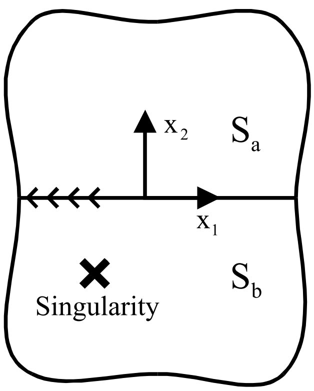

Fig. 3. A singularity in a bimaterial.

singularity located in lower half-space is treated, in which the elastic constants of material b are implied in $\Phi_0(z)$ and $\Omega_0(z)$. The complex potentials are assumed to be

$$\Phi(z) = \begin{cases} \Phi_0(z) + \Phi_a(z), & \text{in } S_a, \\ \Phi_0(z) + \Phi_b(z), & \text{in } S_b, \end{cases} \qquad \Omega(z) = \begin{cases} \Omega_0(z) + \Omega_a(z), & \text{in } S_a, \\ \Omega_0(z) + \Omega_b(z), & \text{in } S_b, \end{cases} \tag{9}$$

where S_a, the upper half-space, and S_b, the lower half-space, are occupied by material a and b, respectively. The continuity of tractions and displacements across the interface is used to determine $\Phi_a(z)$, $\Phi_b(z)$, $\Omega_a(z)$, and $\Omega_b(z)$ analytic in their given regions with the argument of the analytic continuation properly employed. The procedure is identical to that of Suo (1989), details of which are suppressed here, and the results are as follows:

$$\Phi(z) = \begin{cases} (1 + \Lambda_{ab})\Phi_0(z), & \text{in } S_a, \\ \Phi_0(z) + \Pi_{ab}\bar{\Omega}_0(z), & \text{in } S_b, \end{cases} \tag{10a}$$

$$\Omega(z) = \begin{cases} (1 + \Pi_{ab})\Omega_0(z), & \text{in } S_a, \\ \Omega_0(z) + \Lambda_{ab}\bar{\Phi}_0(z), & \text{in } S_b, \end{cases} \tag{10b}$$

where Λ_{ab} and Π_{ab} are as defined in Eq. (5).

As special cases of Eq. (9), a singularity in the lower half-space with free or rigid surface can be dealt with. For the former case, $\Lambda_{ab} = \Pi_{ab} = -1$ and therefore

$$\Phi(z) = \Phi_0(z) - \bar{\Omega}_0(z), \quad \Omega(z) = \Omega_0(z) - \bar{\Phi}_0(z), \quad \text{in } S_b, \tag{11}$$

while for the latter case, $\Lambda_{ab} = 1/\Pi_{ab} = \kappa_b$ and therefore

$$\Phi(z) = \Phi_0(z) + \bar{\Omega}_0(z)/\kappa_b, \quad \Omega(z) = \Omega_0(z) + \kappa_b\bar{\Phi}_0(z), \quad \text{in } S_b. \tag{12}$$

For a singularity located in upper half-space, the solution is also assumed to be identical to Eq. (9). By the similar procedures used in Eqs. (10a) and (10b), one obtains

$$\Phi(z) = \begin{cases} \Phi_0(z) + \Pi_{ba}\bar{\Omega}_0(z), & \text{in } S_a, \\ (1 + \Lambda_{ba})\Phi_0(z), & \text{in } S_b, \end{cases} \tag{13a}$$

$$\Omega(z) = \begin{cases} \Omega_0(z) + \Lambda_{ba}\bar{\Phi}_0(z), & \text{in } S_a, \\ (1 + \Pi_{ba})\Omega_0(z), & \text{in } S_b, \end{cases} \tag{13b}$$

in which the elastic constants of material a are implied in $\Phi_0(z)$ and $\Omega_0(z)$.

5. A singularity in a trimaterial and the alternating technique

To analyze a singularity in a trimaterial with two parallel interfaces as shown in Fig. 4, the alternating technique together with the results of Sections 3 and 4 is applied. The procedure is precisely the same as that of Part I (Choi and Earmme, in press) for the anisotropic case. Since it is difficult to find a solution satisfying all the continuity conditions along two interfaces at the same time, the method of analytic continuation should be applied to two interfaces alternatively. In order to use the method of analytic continuation for the upper interface lying off x_1-axis, we consider a coordinate translation described as below.

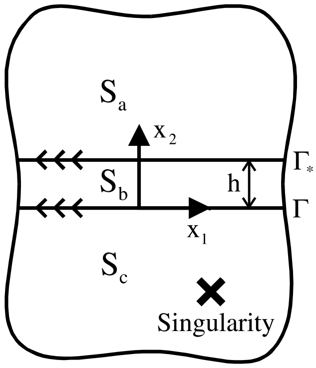

Fig. 4. A singularity in a trimaterial.

5.1. A coordinate translation

Suppose that regions $S_a : x_2 \geqslant h$ and $S_b : x_2 \leqslant h$ occupied by material a and b, respectively, are perfectly bonded along the interface $x_2 = h$. With a coordinate translation $z^* = z - ih$ (see Fig. 4 with material c = material b), let us reformulate the bimaterial solution obtained in the previous section. The potentials $\Phi(z)$ and $\Omega(z)$ in the x_1x_2 coordinate system are related to the potentials $\Phi^*(z^*)$ and $\Omega^*(z^*)$ in the $x_1^*x_2^*$ coordinate system by

$$\Phi(z) = \Phi^*(z^*), \qquad \Omega(z) = \Omega^*(z^*) + 2ih\Phi'^*(z^*). \tag{14}$$

Thus if Eqs. (10a) and (10b) are reinterpreted as the bimaterial solution in the $x_1^*x_2^*$ coordinate system, Eq. (14) leads to

$$\Phi(z) = \Phi^*(z^*) = \begin{cases} (1 + \Lambda_{ab})\Phi_0^*(z^*), & \text{in } S_a, \\ \Phi_0^*(z) + \Pi_{ab}\bar{\Omega}_0^*(z^*), & \text{in } S_b, \end{cases} \tag{15a}$$

$$\Omega(z) = \Omega^*(z^*) + 2ih\Phi'^*(z^*) = \begin{cases} (1 + \Pi_{ab})\Omega_0^*(z^*) + 2ih(1 + \Lambda_{ab})\Phi_0'^*(z^*), & \text{in } S_a, \\ \Omega_0^*(z^*) + \Lambda_{ab}\bar{\Phi}_0^*(z^*) + 2ih[\Phi_0'^*(z^*) + \Pi_{ab}\bar{\Omega}_0'^*(z^*)], & \text{in } S_b, \end{cases} \tag{15b}$$

Substituting the homogeneous solution, $\Phi_0^*(z^*) = \Phi_0(z)$ and $\Omega_0^*(z^*) = \Omega_0(z) - 2ih\Phi_0'(z)$ together with $\bar{\Phi}_0^*(z^*) = \bar{\Phi}_0(z - 2ih)$ and $\bar{\Omega}_0^*(z^*) = \bar{\Omega}_0(z - 2ih) + 2ih\bar{\Phi}_0'(z - 2ih)$ into Eqs. (15a) and (15b), one can find the bimaterial solution in x_1x_2 coordinate system as follows:

$$\Phi(z) = \begin{cases} (1 + \Lambda_{ab})\Phi_0(z), & \text{in } S_a, \\ \Phi_0(z) + \Pi_{ab}\bar{\Omega}_0(z - 2ih) + 2ih\Pi_{ab}\bar{\Phi}_0'(z - 2ih), & \text{in } S_b, \end{cases} \tag{16a}$$

$$\Omega(z) = \begin{cases} (1 + \Pi_{ab})\Omega_0(z) + 2ih(\Lambda_{ab} - \Pi_{ab})\Phi_0'(z), & \text{in } S_a, \\ \Omega_0(z) + \Lambda_{ab}\bar{\Phi}_0(z - 2ih) + 2ih\Pi_{ab}\bar{\Omega}_0'(z - 2ih) - 4h^2\Pi_{ab}\bar{\Phi}_0''(z - 2ih), & \text{in } S_b. \end{cases} \tag{16b}$$

5.2. Case I: A singularity embedded in S_c

With the aid of the result of the coordinate translation described in Section 5.1, now the problem in Fig. 4 is considered, in which material a, b and c occupying regions $S_a : x_2 \geqslant h$, $S_b : h \geqslant x_2 \geqslant 0$, and $S_c : x_2 \leqslant 0$, respectively, are perfectly bonded along two parallel interfaces $\Gamma : x_2 = 0$ and $\Gamma_* : x_2 = h$. Let the potentials for the three regions be

$$\Phi(z) = \begin{cases} \sum_{n=1}^{\infty} \Phi_{\mathrm{a}n}(z), & \text{in } S_{\mathrm{a}}, \\ \sum_{n=1}^{\infty} \Phi_n(z) + \sum_{n=1}^{\infty} \Phi_{\mathrm{b}n}(z), & \text{in } S_{\mathrm{b}}, \\ \Phi_0(z) + \Phi_{\mathrm{c}0}(z) + \sum_{n=1}^{\infty} \Phi_{\mathrm{c}n}(z), & \text{in } S_{\mathrm{c}}, \end{cases} \tag{17a}$$

$$\Omega(z) = \begin{cases} \sum_{n=1}^{\infty} \Omega_{\mathrm{a}n}(z), & \text{in } S_{\mathrm{a}}, \\ \sum_{n=1}^{\infty} \Omega_n(z) + \sum_{n=1}^{\infty} \Omega_{\mathrm{b}n}(z), & \text{in } S_{\mathrm{b}}, \\ \Omega_0(z) + \Omega_{\mathrm{c}0}(z) + \sum_{n=1}^{\infty} \Omega_{\mathrm{c}n}(z), & \text{in } S_{\mathrm{c}}. \end{cases} \tag{17b}$$

By applying the method of analytic continuation to two interfaces alternatively, the unknown potentials $\Phi_{\mathrm{c}0}(z)$, $\Phi_n(z)$, $\Phi_{\mathrm{a}n}(z)$, $\Phi_{\mathrm{b}n}(z)$, $\Phi_{\mathrm{c}n}(z)$, $\Omega_{\mathrm{c}0}(z)$, $\Omega_n(z)$, $\Omega_{\mathrm{a}n}(z)$, $\Omega_{\mathrm{b}n}(z)$, and $\Omega_{\mathrm{c}n}(z)$ ($n = 1, 2, 3, \ldots$) analytic in their given regions are expressed in terms of $\Phi_0(z)$ and $\Omega_0(z)$, in which the elastic constants of material c are implied. The procedure is similar to the four steps described in Appendix A of Part I (Choi and Earmme, in press), therefore the details are suppressed here. The results are as follows:

$$\Phi(z) = \begin{cases} (1 + \Lambda_{\mathrm{ab}}) \sum_{n=1}^{\infty} \Phi_n(z), & \text{in } S_{\mathrm{a}}, \\ \sum_{n=1}^{\infty} \left[\Phi_n(z) + \Lambda_{\mathrm{cb}}^{-1} \bar{\Omega}_{n+1}(z) \right], & \text{in } S_{\mathrm{b}}, \\ \Phi_0(z) + \Pi_{\mathrm{bc}} \bar{\Omega}_0(z) + (1 + \Lambda_{\mathrm{cb}}^{-1}) \sum_{n=1}^{\infty} \bar{\Omega}_{n+1}(z), & \text{in } S_{\mathrm{c}}, \end{cases} \tag{18a}$$

$$\Omega(z) = \begin{cases} (1 + \Pi_{\mathrm{ab}}) \sum_{n=1}^{\infty} \Omega_n(z) + 2\mathrm{i}h(\Lambda_{\mathrm{ab}} - \Pi_{\mathrm{ab}}) \sum_{n=1}^{\infty} \Phi_n'(z), & \text{in } S_{\mathrm{a}}, \\ \sum_{n=1}^{\infty} \left[\Omega_n(z) + \Pi_{\mathrm{cb}}^{-1} \bar{\Phi}_{n+1}(z) \right], & \text{in } S_{\mathrm{b}}, \\ \Omega_0(z) + \Lambda_{\mathrm{bc}} \bar{\Phi}_0(z) + (1 + \Pi_{\mathrm{cb}}^{-1}) \sum_{n=1}^{\infty} \bar{\Phi}_{n+1}(z), & \text{in } S_{\mathrm{c}}, \end{cases} \tag{18b}$$

where the recurrence formulae for $\Phi_n(z)$ and $\Omega_n(z)$ respectively are

$$\Phi_{n+1}(z) = \begin{cases} (1 + \Lambda_{\mathrm{bc}}) \Phi_0(\mathrm{z}), & \text{for } n = 0, \\ \Pi_{\mathrm{cb}} \left[\Lambda_{\mathrm{ab}} \Phi_n(z + 2\mathrm{i}h) - 2\mathrm{i}h \Pi_{\mathrm{ab}} \Omega_n'(z + 2\mathrm{i}h) - 4h^2 \Pi_{\mathrm{ab}} \Phi_n''(z + 2\mathrm{i}h) \right], & \text{for } n = 1, 2, 3, \ldots, \end{cases} \tag{19a}$$

$$\Omega_{n+1}(z) = \begin{cases} (1 + \Pi_{\mathrm{bc}}) \Omega_0(z), & \text{for } n = 0, \\ \Pi_{\mathrm{ab}} \Lambda_{\mathrm{cb}} \left[\Omega_n(z + 2\mathrm{i}h) - 2\mathrm{i}h \Phi_n'(z + 2\mathrm{i}h) \right], & \text{for } n = 1, 2, 3, \ldots. \end{cases} \tag{19b}$$

Eqs. (18a) and (18b) with Eqs. (19a) and (19b) are the complete solution for the singularity in region S_{c} of the trimaterial.

5.3. Case II: A singularity embedded in S_b

By the same arguments as case I, the other case in which the singularity is located in region S_{b} has the following solution:

$$\Phi(z) = \begin{cases} (1 + \Lambda_{\mathrm{ab}}) \sum_{n=1}^{\infty} \Phi_n(z), & \text{in } S_{\mathrm{a}}, \\ \sum_{n=1}^{\infty} \left[\Phi_n(z) + \Lambda_{\mathrm{cb}}^{-1} \bar{\Omega}_{n+1}(z) \right], & \text{in } S_{\mathrm{b}}, \\ (1 + \Lambda_{\mathrm{cb}}) \Phi_0(z) + (1 + \Lambda_{\mathrm{cb}}^{-1}) \sum_{n=1}^{\infty} \bar{\Omega}_{n+1}(z), & \text{in } S_{\mathrm{c}}, \end{cases} \tag{20a}$$

$$\Omega(z) = \begin{cases} (1+\Pi_{\rm ab})\sum_{n=1}^{\infty}\Omega_n(z) + 2{\rm i}h(\Lambda_{\rm ab}-\Pi_{\rm ab})\sum_{n=1}^{\infty}\Phi_n'(z), & \text{in } S_{\rm a}, \\ \sum_{n=1}^{\infty}\left[\Omega_n(z) + \Pi_{\rm cb}^{-1}\bar{\Phi}_{n+1}(z)\right], & \text{in } S_{\rm b}, \\ (1+\Pi_{\rm cb})\Omega_0(z) + (1+\Pi_{\rm cb}^{-1})\sum_{n=1}^{\infty}\bar{\Phi}_{n+1}(z), & \text{in } S_{\rm c}, \end{cases} \tag{20b}$$

in which the recurrence formulae for $\Phi_n(z)$ and $\Omega_n(z)$ are

$$\Phi_{n+1}(z) = \begin{cases} \Phi_0(z) + \Pi_{\rm cb}\bar{\Omega}_0(z), & \text{for } n=0, \\ \Pi_{\rm cb}\left[\Lambda_{\rm ab}\Phi_n(z+2{\rm i}h) - 2{\rm i}h\Pi_{\rm ab}\Omega_n'(z+2{\rm i}h) - 4h^2\Pi_{\rm ab}\Phi_n''(z+2{\rm i}h)\right], & \text{for } n=1,2,3,\ldots \end{cases} \tag{21a}$$

$$\Omega_{n+1}(z) = \begin{cases} \Omega_0(z) + \Lambda_{\rm cb}\bar{\Phi}_0(z), & \text{for } n=0, \\ \Pi_{\rm ab}\Lambda_{\rm cb}\left[\Omega_n(z+2{\rm i}h) - 2{\rm i}h\Phi_n'(z+2{\rm i}h)\right], & \text{for } n=1,2,3,\ldots \end{cases} \tag{21b}$$

Here the elastic constants in $\Phi_0(z)$ and $\Omega_0(z)$ are for material b. Eqs. (20a) and (20b) with Eqs. (21a) and (21b) are the complete solution for the singularity in region $S_{\rm c}$ of the trimaterial.

6. Discussion

6.1. Convergence and convergence rate of the series solutions

Since it is known that a series solution obtained via the alternating technique converges to the true solution for isotropic elastic materials (Sokolnikoff, 1956), we now discuss the rate of convergence. It is worth pointing out that Eqs. (18a), (18b), (20a) and (20b) are expressed in terms of $\Phi_n(z)$ and $\Omega_n(z)$ ($n = 0, 1, 2, \ldots$), which may be calculated from a homogeneous solution $\Phi_0(z)$ and $\Omega_0(z)$ by the recurrence formulae (19a), (19b), (21a) and (21b). The rate of the convergence depends on the ratios $|\Phi_{n+1}(z)|/|\Phi_n(z)|$ and $|\Omega_{n+1}(z)|/|\Omega_n(z)|$, which in turn depend on the non-dimensional bimaterial constants $\Lambda_{\rm ab}$, $\Lambda_{\rm cb}$, $\Pi_{\rm ab}$, and $\Pi_{\rm cb}$, of which typical values are illustrated in Fig. 2. The smaller the differences of elastic constants of two adjacent materials a and b (or c and b) are, the smaller the magnitudes of $\Lambda_{\rm ab}$ and $\Pi_{\rm ab}$ (or $\Lambda_{\rm cb}$ and $\Pi_{\rm cb}$) are, which is obvious from the definition of $\Lambda_{\rm ab}$ and $\Pi_{\rm ab}$, Eq. (5). Consequently, the convergence rate becomes more rapid as the differences of the elastic constants of the neighboring materials get smaller. For most combinations of materials, Λ and Π are less than 1 and 0.5, respectively (see Fig. 2), which guarantees rapid convergence. It is found that the sum of the first three or four terms provides a good approximation for most combinations of materials, which is verified in an example in Section 6.3. The thickness h of material b also affects the rate of convergence in such a way that as h gets larger, the series solution is more rapidly convergent, because the ordinates of the image singularities are linearly proportional to h.

Even though materials a and/or c are rigid or non-existent, the solutions still remain valid. For these limiting cases, we replace $\Lambda_{\rm ab}$, $\Lambda_{\rm cb}$, $\Pi_{\rm ab}$, and $\Pi_{\rm cb}$ in the solution (18a)–(21b) by those indicated in Table 1 for the four special combinations of three dissimilar materials. All the combinations illustrated in Table 1 are meaningful for a singularity located in $S_{\rm b}$, while only the combinations 3 and 4 have the meaning for a singularity located in $S_{\rm c}$. For another limiting case in which two adjacent materials, say materials a and b,

Table 1
Special combinations of three dissimilar materials forming a trimaterial

Combination type	1	2	3	4
Material a	Empty	Rigid	Empty	Rigid
Material b	Elastic	Elastic	Elastic	Elastic
Material c	Empty	Rigid	Elastic	Elastic
Λ_{ab}	-1	κ_b	-1	κ_b
Π_{ab}	-1	$1/\kappa_b$	-1	$1/\kappa_b$
Λ_{cb}	-1	κ_b	Λ_{cb}	Λ_{cb}
Π_{cb}	-1	$1/\kappa_b$	Π_{cb}	Π_{cb}

are identical, the series solution for a trimaterial reduces to the bimaterial one. Furthermore, if material b is non-existent or rigid, the trimaterial solutions (18a) and (18b) with solutions (19a) and (19b) reduces to the solution (11) or solution (12), respectively.

6.2. The energetic forces exerted on a dislocation

The energetic force on a dislocation segment is given by (Peach and Koehler, 1950)

$$\mathrm{d}\boldsymbol{f} = (\sigma \cdot \mathbf{b}) \times \mathrm{d}\mathbf{l}, \tag{22}$$

for the stress field σ, the Burgers vector $\mathbf{b}$ and the line segment $\mathrm{d}\mathbf{l}$. Note that the trimaterial solution for a dislocation consists of a singular term and the other regular terms corresponding to the image singularities. Using Eqs. (18a), (18b), (or Eqs. (20a) and (20b)), (1), (2) and (22), the image forces in x_2 direction per unit length of a dislocation due to two parallel interfaces in a trimaterial are given by

$$f_2 = -\mathrm{Re}\left\{(b_1 + \mathrm{i}b_2)\left\{\Pi_{cb}\left[2\bar{\Omega}_0(s) + \Omega_0(\bar{s}) + (s-\bar{s})\bar{\Omega}_0'(s)\right] - \Lambda_{cb}\bar{\Phi}_0(s) + \sum_{n=2}^{\infty}\left[2\Phi_n(s) + \bar{\Phi}_n(\bar{s}) + (s-\bar{s})\Phi_n'(s)\right] - \sum_{n=2}^{\infty}\Omega_n(s) + \Lambda_{cb}^{-1}\sum_{n=2}^{\infty}\left[2\bar{\Omega}_n(s) + \Omega_n(\bar{s}) + (s-\bar{s})\bar{\Omega}_n'(s)\right] - \Pi_{cb}^{-1}\sum_{n=2}^{\infty}\bar{\Phi}_n(s)\right\}\right\}, \tag{23}$$

$$f_2 = -\mathrm{Re}\left\{(b_1 + \mathrm{i}b_2)\left\{\Pi_{bc}\left[2\bar{\Omega}_0(s) + \Omega_0(\bar{s}) + (s-\bar{s})\bar{\Omega}_0'(s)\right] - \Lambda_{bc}\bar{\Phi}_0(s) + (1+\Lambda_{cb}^{-1})\sum_{n=2}^{\infty}\left[2\bar{\Omega}_n(s) + \Omega_n(\bar{s}) + (s-\bar{s})\bar{\Omega}_n'(s)\right] - (1+\Pi_{cb}^{-1})\sum_{n=2}^{\infty}\bar{\Phi}_n(s)\right\}\right\} \tag{24}$$

for a dislocation in material b or c, respectively. It is obvious that the image force f_1 in the x_1 direction is equal to zero. The stress field may originate not only from the image field required to satisfy the boundary conditions, but also from external sources such as the other dislocations, residual stresses, applied forces, etc. One may also evaluate the image forces due to the other external agencies in the same way.

6.3. Example: film/substrate structure with a dislocation

We revisit the film/substrate structure with an edge dislocation as shown in Fig. 5, which was previously solved by Lee and Dundurs (1973) and Zhang (1995) by employing the Fourier transform technique. The

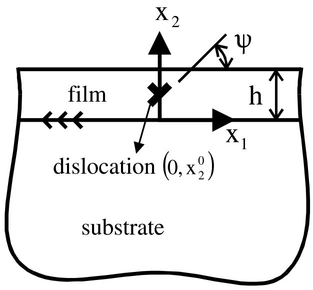

Fig. 5. A dislocation in a film on a substrate.

slip plane of an edge dislocation in the film is considered to be inclined at an angle ψ with respect to the interface plane. The Burgers vector of the edge dislocation is given by $\mathbf{b} = b(\cos\psi, \sin\psi, 0)$ and the tangent line is along the x_3 direction. The image forces f_2/f_{free} exerted on the dislocation with $\psi = 0°$ and 90° are plotted for various values of Dundurs parameters in Figs. 6 and 7, respectively, in which the curves are evaluated with terms up to $n = 4$ in Eq. (23), and the normalizing constant f_{free} is the image force on a dislocation at a distance h from the free surface in half space. It is found that the contributions of terms with $n = 2$, 3, and 4 to the image forces are approximately 16%, 3%, and 0.7%, respectively. It is likely that the error of approximations with terms up to $n = 4$ is <1%. Present result (Figs. 6 and 7) agrees well with that of Lee and Dundurs (1973).

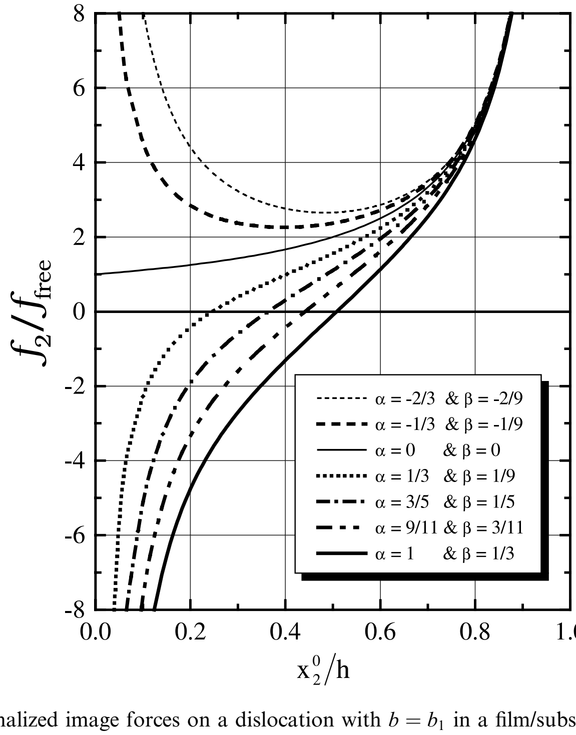

Fig. 6. Normalized image forces on a dislocation with $b = b_1$ in a film/substrate structure.

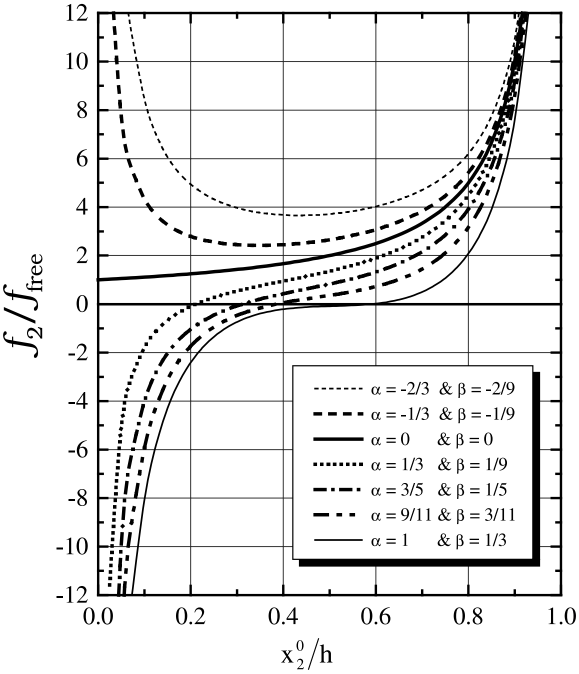

Fig. 7. Normalized image forces on a dislocation with $b = b_2$ in a film/substrate structure.

7. Conclusion

The alternating technique and the method of analytic continuation are employed to study the singularities in an isotropic trimaterial. A homogeneous solution for singularities serves as a base to derive the trimaterial solution for the same singularities in a series form. The convergence rate of the series solution depends on the material combinations and the thickness of middle material. The smaller the mismatch of elastic constants of adjacent materials is, the more rapid the convergence rate is. In the limiting cases, in which one material (or even two materials) in an isotropic trimaterial is rigid or non-existent, the solution still remains valid. Furthermore, as two adjacent materials degenerates to be a homogeneous one, the trimaterial solution reduces to the bimaterial one. Consequently, the trimaterial solution studied here can be applied to a variety of problems, e.g. a bimaterial (including a half-plane problem), a finite thin film on semi-infinite substrate, and a finite strip of thin film, etc. In fact, the merit of this trimaterial solution is its wide applicability to bimaterial problems in addition to the trimaterial problem per se. Even though there are some known solution procedures for the bimaterial problem using the Fourier transform, the present study is much simpler and straightforward.

Acknowledgements

Financial support from KOSEF (The Korea Science and Engineering Foundation) through CEPM (Center for Electronic Packaging Materials) at KAIST is gratefully acknowledged.

Appendix A. The corresponding anti-plane problem in an isotropic trimaterial

Consider an anti-plane singularity in an isotropic trimaterial. The relations between the stresses and the non-zero displacement $u_3 = u_3(x_1, x_2)$ are given as

$$\sigma_{3\alpha} = 2G\frac{\partial u_3}{\partial x_\alpha}, \quad \alpha = 1, 2, \tag{A.1}$$

where G is the shear modulus. Substituting Eq. (A.1) into the equilibrium equation, we obtain the Laplace equation

$$\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} = 0. \tag{A.2}$$

Therefore, we can take the solution of Eq. (A.2) for the imaginary part of an analytic function $w(z)$ of a complex variable $z = x_1 + \mathrm{i}x_2$, i.e.,

$$u_3 = \frac{1}{2\mathrm{i}}[w(z) - \bar{w}(\bar{z})]. \tag{A.3}$$

Combining Eqs. (A.1) and (A.3), one can write

$$\sigma_{32} + \mathrm{i}\sigma_{31} = Gw'(z). \tag{A.4}$$

Then, the solution to any anti-plane deformation has been resolved to finding a proper function $w(z)$, which satisfies given boundary conditions.

First, the solution of a singularity, a line force or a screw dislocation, in an infinite homogeneous medium is written as

$$w_0(z) = \frac{1}{2\pi}\left(b_3 + \frac{p_3}{G\mathrm{i}}\right)\ln(z - s), \tag{A.5}$$

where b_3 is the Burgers vector of a screw dislocation and p_3 the magnitude of a line force per unit length. By the method of analytic continuation, the same singularity embedded in an isotropic bimaterial as shown in Fig. 3 may be expressed as

$$w(z) = \begin{cases} (1 + t_{\mathrm{ab}})w_0(z), & \text{in } S_{\mathrm{a}}, \\ w_0(z) + t_{\mathrm{ab}}\bar{w}_0(z), & \text{in } S_{\mathrm{b}}, \end{cases} \tag{A.6}$$

where $t_{\mathrm{ab}} = (G_{\mathrm{a}} - G_{\mathrm{b}})/(G_{\mathrm{a}} + G_{\mathrm{b}})$. The result implies that the solution of a singularity in a bimaterial is constructed by the proper arrangement of image singularities. In other words, in region S_{a} the solution consists of an image singularity at $s = x_1^0 + \mathrm{i}x_2^0$ ($x_2^0 < 0$) with the strength multiplied by $(1 + t_{\mathrm{ab}})$, while in region S_{b} the solution is made up of the given singularity at s and an image singularity at $\bar{s}$ with the strength multiplied by t_{ab}.

By applying the alternating technique, the solution of the same singularity embedded in an isotropic trimaterial as shown in Fig. 4 are obtained as

$$w(z) = \begin{cases} (1 + t_{\mathrm{ab}})\sum_{n=0}^{\infty} w_n(z), & \text{in } S_{\mathrm{a}}, \\ \sum_{n=0}^{\infty}\left[w_n(z) + t_{\mathrm{cb}}^{-1}\bar{w}_{n+1}(z)\right], & \text{in } S_{\mathrm{b}}, \\ w_0(z) + t_{\mathrm{bc}}\bar{w}_0(z) + (1 + t_{\mathrm{cb}})\sum_{n=0}^{\infty} t_{\mathrm{cb}}^{-1}\bar{w}_{n+1}(z), & \text{in } S_{\mathrm{c}} \end{cases} \tag{A.7}$$

for a singularity in S_{c}, where the recurrence formula of $w_n(z)$ is

$$w_n(z) = (t_{\mathrm{ab}}t_{\mathrm{cb}})^n(1 + t_{\mathrm{bc}})w_0(z + 2hn\mathrm{i}), \quad n = 0, 1, 2, \ldots, \tag{A.8}$$

and

$$w(z) = \begin{cases} (1+t_{ab})\sum_{n=0}^{\infty} w_n(z), & \text{in } S_a, \\ \sum_{n=0}^{\infty}\left[w_n(z) + t_{cb}^{-1}\bar{w}_{n+1}(z)\right], & \text{in } S_b, \\ (1+t_{cb})\left[w_0(z) + \sum_{n=0}^{\infty} t_{cb}^{-1}\bar{w}_{n+1}(z)\right], & \text{in } S_c \end{cases} \tag{A.9}$$

for a singularity in S_b, in which the recurrence formula of $w_n(z)$ is

$$w_n(z) = (t_{ab}t_{cb})^n[w_0(z+2hn\mathrm{i}) + t_{cb}\bar{w}_0(z+2hn\mathrm{i})], \quad n = 0, 1, 2, \ldots. \tag{A.10}$$

Eq. (A.9) with Eq. (A.10) for a singularity in S_b is identical to the result of Chao and Kao (1997) obtained by iterations of Möbius transformation. The above results (A.7)–(A.10) show that the solution of a singularity in a trimaterial can be constructed by the proper arrangement of an infinite number of image singularities. For example, the solution for region S_a in Eq. (A.7) is made up of image singularities located in $s - 2hn\mathrm{i}$ $(n = 0, 1, 2, \ldots)$ with the strength multiplied by $(1+t_{ab})(t_{ab}t_{cb})^n(1+t_{bc})$ $(n = 0, 1, 2, \ldots)$, respectively. One may prove that the series solutions (A.7)–(A.10) are uniformly convergent because of $|t_{ab}t_{cb}| < 1$ for all the material combinations, except for the special case $|t_{ab}t_{cb}| = 1$.

References

Chao, C.K., Kao, B., 1997. A thin cracked layer bonded to an elastic half-space under an antiplane concentrated load. Int. J. Fract. 83, 223–241.

Choi, S.T., Earmme, Y.Y. Elastic study on singularities interacting with interfaces using alternating technique: Part I Anisotropic trimaterial, Int. J. Solids Struct. 39, 943–957.

Dundurs, J., 1969. Discussion: Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading. ASME J. Appl. Mech. 36, 650–652.

Erdogan, F., Wu, B., 1993. Interface crack problems in layered orthotropic materials. J. Mech. Phys. Solids 41, 889–917.

Fleck, N.A., Hutchinson, J.W., Suo, Z., 1991. Crack path selection in a brittle adhesive layer. Int. J. Solids Struct. 27, 1683–1703.

Freund, L.B., 2000. The mechanics of electronic materials. Int. J. Solids Struct. 37, 185–196.

Freund, L.B., 1993. The mechanics of dislocations in strained-layer semiconductor materials. In: Hutchinson, J.W., Wu, T.Y. (Eds.), Advances in Applied Mechanics, vol. 30. Academic Press, New York.

Lee, M.S., Dundurs, J., 1973. Edge dislocation in a surface layer. Int. J. Engng Sci. 11, 87–94.

Muskhelishvili, N.I., 1953. Some Basic Problems of the Mathematical Theory of Elasticity. Noordhoff, Groningen, pp. 105–175.

Peach, M.O., Koehler, J.S., 1950. The forces exerted on dislocations and the stress field produced by them. Phys. Rev. II 80, 436–439.

Sokolnikoff, I.S., 1956. Mathematical Theory of Elasticity. McGraw-Hill, New York, pp. 318–326.

Suga, T., Elssner, G., Schmauder, S., 1988. Composite parameters and mechanical compatibility of material joints. J. Comp. Mater. 22, 917–934.

Suo, Z., 1989. Singularities interacting with interfaces and cracks. Int. J. Solids Struct. 25, 1133–1142.

Suo, Z., Hutchinson, J.W., 1989a. Sandwich test specimens for measuring interface crack toughness. J. Mater. Sci. Engng. A 107, 135–143.

Suo, Z., Hutchinson, J.W., 1989b. Steady-state cracking in brittle substrate beneath adherent films. Int. J. Solids Struct. 25, 1337–1353.

Zhang, T.Y., 1995. Effect of elastic constants on the critical thickness of an epilayer. Phys. Stat. Sol. (a) 152, 415–429.